Week 4 Worksheet Thursday

Instructions. Discuss with your group mates and do the following problems. You are not expected to finish all the problems. :)

Inform your TA if you find any error in the Solutions. !!

Today's Main Topic:
Trig limit and Sandwich theorem

1. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$

$\downarrow 0$ as $x\to 0$ $\qquad$ $\downarrow 0$ as $x\to 0$

$\Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by sandwich thm.

2. $\lim_{x\to 0} x^4\cos\left(\frac{1}{x}\right)$

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$-x^4 \le x^4\cos\left(\frac{1}{x}\right) \le x^4$

$\downarrow 0$ as $x\to 0$ $\downarrow 0$

$\Rightarrow \lim_{x\to 0} x^4\cos\left(\frac{1}{x}\right) = 0$ by sandwich thm.

3. $\lim_{x\to\infty} \frac{1+\sin^2\left(\sqrt{x^2+49}\right)}{\pi+x^2}$

$-1 \le \sin\left(\sqrt{x^2+49}\right) \le 1$

$0 \le \sin^2\left(\sqrt{x^2+49}\right) \le 1$

$1 \le 1+\sin^2\left(\sqrt{x^2+49}\right) \le 2$

$\frac{1}{\pi+x^2} \le \frac{1+\sin^2\sqrt{x^2+49}}{\pi+x^2} \le \frac{2}{\pi+x^2}$

$\downarrow 0$ as $x\to\infty$ $\downarrow 0$

$\Rightarrow ? = 0$ by sandwich thm.

4. $\lim_{t\to 1} 1+(t-1)^{23}\cos\left(\frac{35}{1-\sqrt{t}}\right)$ *(2013 Spring)*

$-1 \le \cos\left(\frac{35}{1-\sqrt{t}}\right) \le 1$

$-\left|(t-1)^{23}\right| \le (t-1)^{23}\cos\left(\frac{35}{1-\sqrt{t}}\right) \le \left|(t-1)^{23}\right|$

$\downarrow 0$ as $t\to 1$ $\downarrow 0$

$\Rightarrow ? = 1+0 = 1$ by sandwich thm

5. $\lim_{x\to\infty} \frac{\sin x}{x} + \sqrt{x^4+x^2+1} - x^2$ *(2012 fall)*

$-1 \le \sin x \le 1$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$\downarrow 0$ as $x\to\infty$ $\downarrow 0$

By sandwich thm, $\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$

$\lim_{x\to\infty} \sqrt{x^4+x^2+1} - x^2$

$= \lim_{x\to\infty} \frac{x^4+x^2+1-x^4}{\sqrt{x^4+x^2+1}+x^2}$

$= \lim_{x\to\infty} \frac{x^2+1}{\sqrt{x^4+x^2+1}+x^2} \cdot \frac{1/x^2}{1/x^2}$

$= \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{\sqrt{1+\frac{1}{x^2}+\frac{1}{x^4}}+1} = \frac{1}{2}$

$\Rightarrow ? = 0 + \frac{1}{2} = \frac{1}{2}$

6. $\lim_{t\to 0} \frac{t}{\sin t}$

Since $\lim_{t\to 0} \frac{\sin t}{t} = 1$,

$\lim_{t\to 0} \frac{t}{\sin t} = 1$

[Why? It's just the reciprocal!]

★ $\boxed{\cos 0 = 1}$

Problems on this page will be covered in next discussion. No worries about them for now! ☺

7. $\lim_{x\to 0} \frac{\tan x}{x}$

$= \lim_{x\to 0} \frac{\sin x}{\cos x} \frac{1}{x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \frac{1}{\cos x}$

$= 1 \cdot 1$

$= 1$

8. $\lim_{\theta\to 0} \frac{\sin(3\theta)}{5\theta}$ *(similar to 2014 and 2015 exams)*

$= \lim_{\theta\to 0} \frac{\sin(3\theta)}{3\theta} \frac{3\theta}{5\theta}$

$= 1 \cdot \frac{3}{5}$

$= \frac{3}{5}$

9. $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

$= \lim_{x\to 0} \frac{\sin(3x)}{3x} \frac{2x}{\sin(2x)} \cdot \frac{3x}{2x}$

$= 1 \cdot 1 \cdot \frac{3}{2}$

$= \frac{3}{2}$

10. $\lim_{x\to 0} \frac{\tan(4x)}{\sin(3x)}$

$= \lim_{x\to 0} \frac{\sin(4x)}{\cos(4x)} \frac{1}{\sin(3x)}$

$= \lim_{x\to 0} \frac{\sin(4x)}{4x} \frac{1}{\cos(4x)} \frac{3x}{\sin(3x)} \frac{4x}{3x}$

$= 1 \cdot 1 \cdot 1 \cdot \frac{4}{3}$

$= \frac{4}{3}$

11. $\lim_{x\to 0} \frac{\sin(3x)}{x(x+2)}$ *(2014 fall)*

$= \lim_{x\to 0} \frac{\sin(3x)}{3x} \frac{3x}{x(x+2)}$

$= 1 \cdot \frac{3}{2}$

$= \frac{3}{2}$

12. $\lim_{x\to 0} \frac{x + x\cos x}{\sin x \cos x}$ *(2015 fall)*

$= \lim_{x\to 0} \frac{x}{\sin x} \frac{x + x\cos x}{\cos x} \frac{1}{x}$ ($\frac{x}{\sin x} \to 1$)

$= \lim_{x\to 0} \frac{1 + \cos x}{\cos x}$

$= \frac{1+1}{1} = 2$